

INVARIANCE OF BIFURCATION EQUATIONS FOR HIGH DEGENERACY BIFURCATIONS OF NON-AUTONOMOUS PERIODIC MAPS

HENRIQUE M. OLIVEIRA

ABSTRACT. Bifurcations of the A_μ class in Arnold's classification, in non-autonomous p -periodic difference equations generated by parameter depending families with p maps, are studied. It is proved that the conditions of degeneracy, non-degeneracy and unfolding are invariant relative to cyclic order of compositions for any natural number μ . The main tool for the proofs is local topological conjugacy. Invariance results are essential to the proper definition of the bifurcations of class A_μ , and lower codimension bifurcations associated, using all the possible cyclic compositions of the fiber families of maps of the p -periodic difference equation. Finally, we present two actual examples of class A_3 or swallowtail bifurcation occurring in period two difference equations for which the bifurcation conditions are invariant.

1. INTRODUCTION

1.1. Motivation. Some works on bifurcation theory of non-autonomous dynamical systems emerged recently, [11, 28]. There are some difficulties to overcome in non-autonomous systems, both with continuous or discrete time. As a starting point it is necessary to set a proper definition of dynamical system [6, 12, 22] and of attractor and repeller [5]. It is also necessary to define clearly the concept of bifurcation. There is a good set of research works on this subject such as [2, 16, 19, 20, 21, 22, 26, 27, 28, 29, 30, 31].

In this paper we are concerned with the definition of bifurcation equations for local bifurcations in one-dimensional p -periodic maps or p -periodic difference equations. In particular, we focus our attention on the A_μ class of bifurcations in the classification of Arnold [3, 4] for the positive integer μ . The main result of the paper is to prove the invariance of the A_μ bifurcation conditions relative to the cyclic order of the maps in the iteration. Actually, we prove all the results for alternating maps, i.e., with $p = 2$ or two fiber maps and for fixed points of the composition maps. This approach has the advantage of being simple in presentation, notation and comfortable to the reader compared to the direct study of p compositions and general k -periodic orbits. The generalization to periodic orbits of p -periodic maps is carried after, being only an exercise of composition and repeated application of the methods developed for alternating maps.

The A_μ class of bifurcation in the autonomous case occurs when one has one real dynamic variable x , the parameter space is real μ -dimensional and the related family of mappings satisfy a set of degeneracy conditions. These conditions provide

Date: February, 2015 - AMS: Primary:37G15; Secondary: 39A28.

Key words and phrases. Topological conjugacy, A_μ degenerate bifurcation, non-autonomous map, p -periodic map, alternating system.

topological equivalence to the unfolding of the germ [4] $x \pm x^{\mu+1}$ at the origin. Since there are many different approaches in the literature, in this work we follow the definitions of [4] concerning germ, topological equivalence, unfolding, codimension and classification of singularities and bifurcations. We suggest as an introduction to the general subject of bifurcations the book [23]. The A_μ class includes the fold, for $\mu = 1$, the cusp, for $\mu = 2$, the swallowtail, for $\mu = 3$, and the butterfly, for $\mu = 4$ [4, 13, 35, 36].

At the end of this paper we use the equations of the swallowtail bifurcation, i.e., A_3 as an example of our results. In this case the bifurcation set¹ in parameter space is made up of three surfaces of fold bifurcations, which meet in two lines of cusp bifurcations and one line of simultaneous double fold, which in turn meet at a single swallowtail bifurcation point as we can see in Figure 1. This bifurcation has codimension three [23], since one needs three independent parameters to unfold completely the bifurcation.

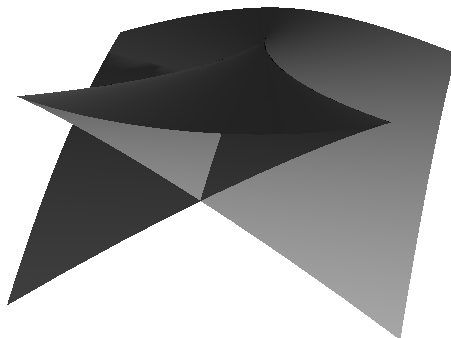


FIGURE 1. The bifurcation set in the parameter space near the origin, the most degenerate point where the A_3 singularity occurs. The control space is real three dimensional. The cut facing the observer is at $a = 1$.

On the subject of codimension see also [4, 8, 14, 17]; we note that the definition of codimension of [14] is different from the one provided by [4] and [23] but the results are basically the same, modulus personal gusto.

The p maps of the family can exhibit a plethora of geometrical behavior not present when we study lower codimension bifurcation. For instance, for $\mu = 3$ and alternating maps, the Schwarzian derivative cannot be simultaneously negative for the two maps at the singularity, as will see in the last section. The negative Schwarzian condition restricts severely the geometry of the families of mappings [10, 33]. Without the negative Schwarzian, we have in the unfolding of this singularity a variety of dynamic phenomena not usually seen in most of the works in one dimensional discrete dynamics [7, 10]. Obviously, in this scenario one does not benefit from Singer's Theorem [34].

¹For bifurcation set see definition 2.2 below.

1.2. Overview. We organized this paper in four sections. In Section 2 we introduce basic concepts including a brief recollection of the A_μ bifurcation equations for families of autonomous real maps.

In Section 3, the core of this work, we study in detail the equations of bifurcation for alternating systems. We prove that when we perform a change in the order of composition of the maps the degeneracy conditions, the non-degeneracy conditions and the transversality conditions remain invariant. These results establish that this type of bifurcation is well defined in the general case of alternating systems. Finally we provide a straightforward generalization of the results on alternating system to periodic orbits of p -periodic maps.

In Section 4 we prove some conditions that override the possibility of A_3 or swallowtail bifurcation in the case of alternating maps, where $p = 2$. Finally, we present two examples concerning alternating maps. These examples show that this class of high degeneracy bifurcations occurs in simple applications without the need of exotic constructions.

2. PRELIMINARIES

2.1. Basic definitions and notation. We define non-autonomous dynamical system using the definitions of [22]. Consider the non-autonomous iteration given by

$$(1) \quad x_{n+1} = f_n(x_n), \quad x_n \in I_n, \quad \text{with } n \in \mathbb{N},$$

where I_n are real intervals (not necessarily compact and maybe \mathbb{R}) which are the fibers of the non-autonomous dynamical system. The usual distance is defined in I_n . The iteration starts at the initial condition $x_0 \in I_0$. Each map f_n is at least continuous and defined such that

$$\begin{aligned} f_n : I_n &\longrightarrow I_{n+1}, \\ x_n &\longmapsto f_n(x_n) \end{aligned}$$

and $f_n(I_n) \subseteq I_{n+1}$.

The system is periodic of period p if

$$f_{n+p} = f_p \text{ and } f_{n+p}(I_n) \subseteq I_n,$$

for every $n \in \mathbb{N}$, where p is a minimal positive integer. When $p = 2$ with the fibers I_0 and I_1 , such that

$$f_0(I_0) \subseteq I_1 \text{ and } f_1(I_1) \subseteq I_0,$$

we say that we have an *alternating system*.

In the sequel, by $\mathcal{C}(I)$ we denote the collection of all continuous maps in its domain I , by $\mathcal{C}^1(I)$ the collection of all continuously differentiable elements of $\mathcal{C}(I)$ and, in general by $\mathcal{C}^s(I)$, $s \geq 1$, the collection of all elements of $\mathcal{C}(I)$ having continuous derivatives up to order s in I .

Let $f \in \mathcal{C}^1(I)$, and let q be a periodic point of period m . Denoting the derivative by D , q is called a *hyperbolic attractor* if $|Df^m(q)| < 1$, a *hyperbolic repeller* if $|Df^m(q)| > 1$, and *non-hyperbolic* if $|Df^m(q)| = 1$.

Definition 2.1. We say that two continuous maps $f : I \rightarrow I$ and $g : J \rightarrow J$, are topologically conjugate, if there exists a homeomorphism $h : I \rightarrow J$, such that $h \circ f = g \circ h$. We call h the topological conjugacy of f and g .

We use the capital Λ for a vector parameter in \mathbb{R}^μ .

Definition 2.2. *If f_Λ is a family of maps, then the regular values Λ of the parameters are those which have the property that $f_{\tilde{\Lambda}}$ is topologically conjugate to f_Λ for all $\tilde{\Lambda}$ in some open neighbourhood of Λ . If Λ is not a regular value, it is a bifurcation value. The collection of all the bifurcation values is the bifurcation set, $\Omega \subset \mathbb{R}^\mu$, in the parameter space.*

Let f_Λ be a family of maps in $\mathcal{C}^s(I)$. Let Λ_0 be a particular vector parameter and $a \in I$ be a fixed point of f_{Λ_0} , i.e.,

$$a = f_{\Lambda_0}(a),$$

the condition of a being non-hyperbolic is necessary for the existence of a local bifurcation. The existence and nature of that bifurcation depends on other symmetry and differentiable conditions that we will see below. If there exists a local bifurcation we say that (a, Λ_0) is a *bifurcation point* (when there is no risk of confusion, we say that a is a *bifurcation point*).

Notation 2.3. *For notational simplicity we consider the real vector parameter Λ as a standard variable along with the dynamic variable x , i.e., we write*

$$f_\Lambda(x) = f(x, \Lambda),$$

keeping in mind that the compositions are always in the dynamic variable x .

When there are no danger of confusion and no operations regarding the parameter, we denote the evaluation of functions depending on the dynamic variable and the parameter omitting the later, for instance $f_\Lambda(x) = f(x, \Lambda)$ will be denoted by $f(x)$ in order to avoid overload the complicated notation needed for the computations of high order chain rules. Nevertheless, all the maps in this paper depend on the parameter as well on the dynamic variable. We deal with parameter depending families of maps, even when that dependence is not visible in some formulas or expressions.

We denote the derivatives relative to some variable y by ∂_y . Repeated differentiation relative to the same variable is denoted by ∂_{y^n} , for instance $\partial_{yyy} = \partial_{y^3}$. When there is no danger of confusion, we denote strict partial derivatives, i.e., not seeing composed functions, by a subscript. For instance, the third partial derivative of f relative to y is, in that case, denoted by f_{yyy} or f_y^3 .

This means, in particular, that when dealing with the composition of real scalar functions $g(x, t)$ and $f(x, t)$, such that $g \circ f(x, t) = g(f(x, t), t)$, we have the chain rules

$$\begin{aligned} \partial_t g(f(x, t), t) &= g_x(f(x, t), t) f_t(x, t) + g_t(f(x, t), t), \\ \partial_x g(f(x, t), t) &= g_x(f(x, t), t) f_x(x, t). \end{aligned}$$

Along this paper we deal with p -periodic sequences of maps f_0, f_1, \dots, f_{p-1} on a real dynamic variable x and depending on a real vector parameter Λ , such that

$$\begin{aligned} f_j : I_j \times \Theta &\longrightarrow I_{j+1} \\ (x, \Lambda) &\longmapsto f_0(x, \Lambda) \end{aligned}$$

for $j = 0, \dots, p-1$. The fibers I_j for the dynamic variable are intervals of \mathbb{R} and $\Theta \subset \mathbb{R}^\mu$ is the parameter set, $f_j \in \mathcal{C}^{\mu+1}(I_j)$ and $f_j \in \mathcal{C}^1(\Theta)$, with μ a positive integer. Moreover, the property

$$f_j(I_j, \Lambda) \subseteq I_{j+1 \pmod{p}}, \text{ holds for all } \Lambda \in \Theta.$$

In this paper we use the convention that capital letters are used for compositions of maps in the dynamic variable. Capital F and G will be always used for direct and reverse composition of alternating maps

$$F = f_1 \circ f_0 \text{ and } G = f_0 \circ f_1.$$

Consider the set of indexes $j = 0, \dots, p-1$ for the p -periodic system. We set the following notation for the p compositions

$$\begin{aligned} F_0 &= f_{p-1} \circ \dots \circ f_0, \\ F_1 &= f_0 \circ f_{p-1} \circ \dots \circ f_1, \\ &\vdots \\ F_{p-1} &= f_{p-2} \circ \dots \circ f_0 \circ f_{p-1}. \end{aligned}$$

Repeated composition (always in the dynamic variable) is denoted by

$$f^k = \underbrace{(f \circ \dots \circ f)}_k,$$

where k is a positive integer.

2.2. Conditions for the A_μ class of bifurcations in autonomous systems.

In this paragraph, we recall briefly the conditions of class A_μ of local bifurcations in Arnold classification as explained in Theorem of page 20 in Arnold et al. [4]. For the iteration of maps, the normalized germ of the class A_μ is $x \pm x^{\mu+1}$ and has the principal family [4], also called prototype polynomial or normal form [23]

$$x \pm x^{\mu+1} + \lambda_1 + \dots + \lambda_\mu x^{\mu-1},$$

where λ_j , $j = 1, \dots, \mu$, are real parameters.

Giving an autonomous discrete dynamical system generated by the iteration of f , in order to compute the bifurcation points of class A_μ , one has to solve the bifurcation equations [23]

$$(2) \quad \begin{aligned} f(x, \Lambda) &= x, \text{ fixed point equation} \\ f_x(x, \Lambda) &= 1, \text{ non-hyperbolicity condition.} \end{aligned}$$

The simplest of such local bifurcations is the saddle node bifurcation, i.e., A_1 . One assumes, in this case, the generic non-degeneracy condition

$$(3) \quad f_{xx}(x, \lambda) \neq 0$$

and the transversality condition [23]

$$(4) \quad f_\lambda(x, \lambda) \neq 0, \text{ with } \lambda \in \mathbb{R}.$$

We set generically that $\lambda \in \mathbb{R}$, since one needs only one parameter to unfold locally this singularity [1, 4, 8, 14, 15, 23]. The normalized germ of this bifurcation is

$$x \pm x^2,$$

with principal family

$$x \pm x^2 + \lambda,$$

which is weak topologically conjugated to any other family [4, 23] satisfying the bifurcation conditions.

Adding degeneracy conditions, one obtains higher degeneracy local bifurcations.

Therefore, the equations for the occurrence of A_μ class of bifurcations for a general positive integer μ are

$$(5) \quad \begin{aligned} f(x, \Lambda) &= x, \\ f_x(x, \Lambda) &= 1, \\ f_{xx}(x, \Lambda) &= 0, \\ &\vdots \\ f_{x^\mu}(x, \Lambda) &= 0, \end{aligned}$$

with solution (a, Λ_0) . It is easy to see that these conditions are satisfied by the normalized germ $x \pm x^{\mu+1}$ at the origin. One has the non-degeneracy condition

$$(6) \quad f_{x^{\mu+1}}(a, \Lambda_0) \neq 0,$$

for $\mu = 2$ we have the cusp, for $\mu = 3$ the swallowtail and for $\mu = 4$ the butterfly [4, 8, 9, 14, 15, 23]. The transversality condition (see pages 66, 297, 298 and 303 of [23]) at the solution of the above conditions is given by the condition on the non-singularity of the Jacobian matrix of the map $(f, f_x, f_{xx}, \dots, f_{x^{\mu-1}})$ relative to the parameters at the bifurcation point

$$(7) \quad \det \begin{bmatrix} f_{\lambda_1}(a, \Lambda_0) & \cdots & f_{\lambda_\mu}(a, \Lambda_0) \\ \vdots & \ddots & \vdots \\ f_{x^{\mu-1}\lambda_1}(a, \Lambda_0) & \cdots & f_{x^{\mu-1}\lambda_\mu}(a, \Lambda_0) \end{bmatrix} \neq 0,$$

and assures that the vector parameter is enough to unfold the local bifurcation [23]. This happens since condition (7) assures that the μ lower order terms in the Taylor polynomial of f depend uniquely on the μ components of Λ , i.e., $\lambda_1, \dots, \lambda_\mu$.

3. A_μ CLASS OF BIFURCATION IN FAMILIES OF p -PERIODIC MAPS

3.1. Invariance of the bifurcation conditions.

3.1.1. On the invariance of the degeneracy and non-degeneracy conditions for alternating systems. In this paragraph, we study the invariance of the degeneracy conditions of alternating families of maps for all the singularities of class A_μ , using topological conjugacy.

Given an initial condition $x_0 \in I_0$ the alternating system is given by the iteration

$$(8) \quad x_{n+1} = f_{n(\bmod 2)}(x_n, \Lambda), \quad x_n \in I_{n(\bmod 2)}.$$

If there is a pair (a, I_0) , such that after 2 iterations, the iteration returns to (a, I_0) we say that a is a *periodic point in the fiber I_0* with period 2. We note that the point $b = f_0(a, \Lambda)$ is also a periodic point in the fiber I_1 with period 2. Consider the compositions F and G , we have

$$a = F(a, \Lambda) \text{ and } b = G(b, \Lambda).$$

In other words: a (resp. b) is a periodic point with period 2 in fiber I_0 (resp. I_1) of the alternating system (8) iff a (resp. b) is a fixed point of F (resp. G).

These below are the bifurcation equations with $\mu - 1$ degeneracy conditions on derivatives on x stated for F and G which are exactly the same as in the non-autonomous case

$$(9) \quad \begin{cases} F(x, \Lambda) = x, \\ F_x(x, \Lambda) = 1, \\ F_{xx}(x, \Lambda) = 0, \\ \vdots \\ F_{x^\mu}(x, \Lambda) = 0, \end{cases} \quad \text{and} \quad \begin{cases} G(x, \Lambda) = x, \\ G_x(x, \Lambda) = 1, \\ G_{xx}(x, \Lambda) = 0, \\ \vdots \\ G_{x^\mu}(x, \Lambda) = 0. \end{cases}$$

These equations have different solutions for the dynamic variable x , depending on the fiber we choose. At the solutions of (9), the non-degeneracy conditions are

$$(10) \quad F_{x^{\mu+1}}(a, \Lambda_0) \neq 0 \text{ and } G_{x^{\mu+1}}(b, \Lambda_0) \neq 0$$

A natural question arises:

Are the solutions in the parameter space equal for the different compositions F and G ?

The same query was posed in [9, 11] and positively solved in the particular cases dealt in those works for degeneracy conditions until the cusp, i.e., $\mu = 2$.

Indeed, in this paragraph we show that the answer to the question is positive in the general case. We prove that if a parameter vector satisfies equations (9) for F then it is a solution of the system for G .

The next lemma will be used to solve the general problem of the symmetry of the bifurcation equations with respect to the order of composition.

Lemma 3.1. *Let $\mu \geq 1$ and let h and f be real functions satisfying the conditions:*

- (1) *there exists a such that $f(a) = a$ and f is a Lipschitz homeomorphism in some open interval I containing a ;*
- (2) *h is a Lipschitz homeomorphism with Lipschitz constant L in a open neighborhood I_h of a , and there exists an open neighborhood, J_b , of $h(a) = b$ such that its inverse, h^{-1} , is also Lipschitz continuous with Lipschitz constant M ;*
- (3)

$$\lim_{x \rightarrow a} \frac{|f(x) - x|}{|x - a|^\mu} = 0 \text{ and } \lim_{x \rightarrow a} \frac{|f(x) - x|}{|x - a|^{\mu+1}} > 0$$

Then g , the conjugate of f by the homeomorphism h ,

$$g = h \circ f \circ h^{-1}$$

satisfies

$$(11) \quad \lim_{y \rightarrow b} \frac{|g(y) - y|}{|y - b|^\mu} = 0 \text{ and } \lim_{y \rightarrow b} \frac{|g(y) - y|}{|y - b|^{\mu+1}} > 0$$

Proof. We first compute the domain J where y ranges when we compute the limit (11). Of course, we take J to be an open interval containing b such that $J \subseteq h(I_h)$. The limit has meaning if J also satisfies

$$(12) \quad \begin{aligned} h^{-1}(J) &\subseteq I \\ f(h^{-1}(J)) &\subseteq I_h. \end{aligned}$$

As both f and h^{-1} are homeomorphisms we can choose the open interval J small enough just to satisfy the conditions (12). We note that $a \in h^{-1}(J)$ and $a \in f(h^{-1}(J))$.

Let us consider the limit (11). We have

$$\begin{aligned}
0 &\leq \lim_{y \rightarrow b} \frac{|g(y) - y|}{|y - b|^\mu} = \lim_{y \rightarrow b} \frac{|(h \circ f \circ h^{-1})(y) - (h \circ h^{-1})(y)|}{|y - b|^\mu} \\
&\leq L \lim_{y \rightarrow b} \frac{|(f \circ h^{-1})(y) - h^{-1}(y)|}{|y - b|^\mu} \\
&= L \lim_{y \rightarrow b} \frac{|(f \circ h^{-1})(y) - h^{-1}(y)|}{|h^{-1}(y) - a|^\mu} \left(\frac{|h^{-1}(y) - a|}{|y - b|} \right)^\mu \\
&= L \lim_{y \rightarrow b} \frac{|(f \circ h^{-1})(y) - h^{-1}(y)|}{|h^{-1}(y) - a|^\mu} \left(\frac{|h^{-1}(y) - h^{-1}(b)|}{|y - b|} \right)^\mu \\
&\leq LM^\mu \lim_{y \rightarrow b} \frac{|(f \circ h^{-1})(y) - h^{-1}(y)|}{|h^{-1}(y) - a|^\mu}.
\end{aligned}$$

As

$$\lim_{y \rightarrow b} h^{-1}(y) = a,$$

if we set $h^{-1}(y) = x$, it follows

$$0 \leq \lim_{y \rightarrow b} \frac{|g(y) - y|}{|y - b|^\mu} \leq LM^\mu \lim_{x \rightarrow a} \frac{|f(x) - x|}{|x - a|^\mu} = 0.$$

When one has $\lim_{x \rightarrow a} \frac{|f(x) - x|}{|x - a|^{\mu+1}} > 0$, we apply similar reasoning to f to get

$$0 < \lim_{x \rightarrow a} \frac{|f(x) - x|}{|x - a|^{\mu+1}} \leq ML^{\mu+1} \lim_{y \rightarrow b} \frac{|g(y) - y|}{|y - b|^{\mu+1}},$$

therefore

$$\lim_{y \rightarrow b} \frac{|g(y) - y|}{|y - b|^{\mu+1}} > 0,$$

as desired. ■

Using the previous lemma we can easily prove the next result.

Theorem 3.2. *Let $\mu \geq 2$ and let be the alternating family of maps with individual mappings f_0, f_1 with $f_0 \in \mathcal{C}^{\mu+1}(I_0)$, $f_1 \in \mathcal{C}^{\mu+1}(I_1)$ in the dynamic variable and the compositions $F = f_1 \circ f_0$ and $G = f_0 \circ f_1$, satisfying:*

- (1) *There exist a, b , fixed points of F and G respectively*

$$a = F(a, \Lambda),$$

$$b = G(b, \Lambda).$$

- (2) *The non-hyperbolicity condition for F*

$$\partial_x F(x, \Lambda)|_{x=a} = \partial_x f_1(x, \Lambda)|_{x=b} \partial_x f_0(x, \Lambda)|_{x=a} = 1.$$

- (3) *Higher degeneracy conditions for F*

$$\partial_{x^i} F(x, \Lambda)|_{x=a} = 0, \text{ for every } 2 \leq i \leq \mu.$$

- (4) *The non-degeneracy condition for F*

$$\partial_{x^{\mu+1}} F(x, \Lambda)|_{x=a} \neq 0.$$

Then, the composition G , satisfies

$$\partial_{x^i} G(x, \Lambda)|_{x=b} = 0, \text{ for every } 2 \leq i \leq \mu$$

and

$$\partial_{x^{\mu+1}} G(x, \Lambda)|_{x=b} \neq 0.$$

Proof. Properties (1) and (2) imply that f_0, f_1 are diffeomorphisms in suitable neighborhoods of a and b , respectively. Therefore, we can define local inverses. Being local diffeomorphisms, f_0 and f_1 are also local Lipschitz continuous and so their inverses. In particular $f_0(a) = b$ and $f_0^{-1}(b) = a$. We apply Lemma 3.1 to F and G making the identification $f = F$, $g = G$ and $h = f_0$.

By (2) and (3)

$$\lim_{x \rightarrow a} \frac{|F(x) - x|}{|x - a|^\mu} = \lim_{x \rightarrow a} \frac{|(F(x) - x) - (F(a) - a)|}{|x - a|^\mu} = 0.$$

Therefore, $h = f_0$ and F satisfy the hypotheses of Lemma 3.1, and hence the thesis with $G(x) = (f_0 \circ F \circ f_0^{-1})(x)$. Thus, we obtain

$$\lim_{x \rightarrow b} \frac{|(f_0 \circ F \circ f_0^{-1})(x) - x|}{|x - b|^\mu} = \lim_{x \rightarrow b} \frac{|G(x) - x|}{|x - b|^\mu} = 0$$

and

$$\lim_{x \rightarrow b} \frac{|(f_0 \circ F \circ f_0^{-1})(x) - x|}{|x - b|^{\mu+1}} = \lim_{x \rightarrow b} \frac{|G(x) - x|}{|x - b|^{\mu+1}} > 0,$$

that is, the first μ derivatives of $G(x) - x$ are zero at b and the non-degeneracy condition holds as well. ■

3.1.2. Example using Faà di Bruno Formula. Although seeming needless after the previous results, the next example will be important to deduce properties on the geometrical behavior of the composition of maps related to the swallowtail bifurcation at the beginning of Section 4. On the other hand, it is interesting to recover the Faà di Bruno's formula [18, 24], since we will use it to prove the invariance of the transversality conditions. We think that it is possible to establish combinatorial results, using Theorem 3.2 on both ends of the general formula for the derivatives of the compositions. This is an interesting open line of research for readers interested in Bell polynomials and other relevant combinatorial quantities associated with the Faà di Bruno's Formula, see [18, 25, 32].

Example 3.3. (Alternating maps) Giving two real maps f and g defined in real intervals I_0 and I_1 we prove directly that if the second derivative relative to the dynamic variable of any of the two maps $g \circ f$ and $f \circ g$ is zero, then also the other must be zero, disregarding the order of composition. The same holds for the third derivatives. We do this directly, using the chain rule for computing the derivatives of composed maps and its generalization, the Faà di Bruno's Formula [18].

Let f and g be C^3 functions satisfying the conditions:

- (1) $(g \circ f)(a) = a$ and $(f \circ g)(b) = b$, which is $f(a) = b$ and $g(b) = a$.
- (2) $\frac{d(g \circ f)}{dx}(x) \Big|_{x=a} = g'(b) f'(a) = 1$.
- (3) $\frac{d^m(g \circ f)}{dx^m}(x) \Big|_{x=a} = 0$ for $m = 2, 3$.

Let us recall the formula of Faà di Bruno for the derivatives of the composition

$$(13) \quad \frac{d^m (g \circ f)}{dx^m} (x) = m! \sum g^{(n)} (f(x)) \prod_{j=1}^m \frac{1}{\beta_j!} \left(\frac{f^{(j)}(x)}{j!} \right)^{\beta_j},$$

where the sum is over all different solutions β_j in nonnegative integers, β_1, \dots, β_m , of the linear Diophantine equations

$$\sum_{j=1}^m j\beta_j = m, \text{ and } n := \sum_{j=1}^m \beta_j.$$

To avoid to overload this example with indexes we use the notation used in [32]

$$\begin{aligned} f_0 &= f(a), & f_1 &= f'(x)|_{x=a}, \dots, & f_m &= f^{(m)}(x)|_{x=a}, \\ g_0 &= g(b), & g_1 &= g'(x)|_{x=b}, \dots, & g_m &= g^{(m)}(x)|_{x=b}, \\ \frac{d^m (g \circ f)}{dx^m} (x)|_{x=a} &= (gf)_m, & \frac{d^m (f \circ g)}{dx^m} (x)|_{x=b} &= (fg)_m. \end{aligned}$$

With this notation, and taking into account the hypotheses 1, 2 and 3, Faà di Bruno's Formula gives

$$(14) \quad (gf)_m = m! \sum g_n \prod_{j=1}^m \frac{1}{\beta_j!} \left(\frac{f_j}{j!} \right)^{\beta_j}$$

and

$$(15) \quad (fg)_m = m! \sum f_n \prod_{j=1}^m \frac{1}{\beta_j!} \left(\frac{g_j}{j!} \right)^{\beta_j}$$

Condition (2) in this notation is now

$$(16) \quad f_1 g_1 = 1.$$

Let us consider the first two cases: $m = 2$ and $m = 3$, cusp and swallowtail.

Let $m = 2$. We shall use the formula (13), therefore we have to solve the equation

$$\beta_1 + 2\beta_2 = 2,$$

for all possible values of the vector (β_1, β_2) in $\mathbb{N} \times \mathbb{N}$. The only solutions are $(\beta_1, \beta_2) = (0, 1)$, which gives $n = 1$ and $(\beta_1, \beta_2) = (2, 0)$, which gives $n = 2$. So we have

$$\begin{aligned} (17) \quad (gf)_2 &= 2! \left(g_1 \frac{1}{0!} \left(\frac{f_1}{1!} \right)^0 \frac{1}{1!} \left(\frac{f_2}{2!} \right)^1 + g_2 \frac{1}{2!} \left(\frac{f_1}{1!} \right)^2 \frac{1}{0!} \left(\frac{f_2}{2!} \right)^0 \right) = 0 \\ &= g_1 f_2 + g_2 f_1^2 = g_1 f_2 + \frac{g_2}{g_1^2} = 0 \end{aligned}$$

and

$$\begin{aligned} (18) \quad (fg)_2 &= 2! \left(f_1 \frac{1}{0!} \left(\frac{g_1}{1!} \right)^0 \frac{1}{1!} \left(\frac{g_2}{2!} \right)^1 + f_2 \frac{1}{2!} \left(\frac{g_1}{1!} \right)^2 \frac{1}{0!} \left(\frac{g_2}{2!} \right)^0 \right) \\ &= f_1 g_2 + f_2 g_1^2 = \frac{g_2}{g_1} + f_2 g_1^2. \end{aligned}$$

We solve the system with equations (16) and (17) for g_2 , to obtain

$$(19) \quad g_2 = g_2(f_1, f_2) = -\frac{f_2}{f_1^3}.$$

By substituting $g_1 = \frac{1}{f_1}$ and $g_2(f_1, f_2)$ in (18), we get

$$(fg)_2 = -f_1 \frac{f_2}{f_1^3} + f_2 \frac{1}{f_1^2} = 0.$$

Let $m = 3$.

By Faà di Bruno Formula, and taking into account the hypotheses, we obtain

$$(20) \quad \begin{aligned} (gf)_3 &= g_1 f_3 + 3g_2 f_1 f_2 + g_3 f_1^3 \\ &= g_1 f_3 + \frac{3g_2 f_2}{g_1} + \frac{g_3}{g_1^3} = 0. \end{aligned}$$

We solve the system with equations (16), (17) and (20) for g_3 , keeping in mind that g_1 and g_2 have been computed before. Hence, we get

$$(21) \quad g_3 = -\frac{f_3}{f_1^4} - \frac{3\left(-\frac{f_2}{f_1^3}\right)f_2}{f_1^2} = \frac{3f_2^2}{f_1^5} - \frac{f_3}{f_1^4}.$$

By replacing g_1 , g_2 and g_3 by the solutions previously obtained, we find

$$(22) \quad \begin{aligned} (fg)_3 &= f_1 g_3 + 3f_2 g_1 g_2 + f_3 g_1^3 \\ &= f_1 \left(\frac{3f_2^2}{f_1^5} - \frac{f_3}{f_1^4} \right) + 3f_2 \frac{1}{f_1} \left(-\frac{f_2}{f_1^3} \right) + \frac{f_3}{f_1^3} = 0. \end{aligned}$$

3.1.3. On the invariance of the degeneracy and non-degeneracy conditions for periodic orbits of p -periodic systems. What we have just shown in Paragraph 3.1.1 is the invariance of bifurcation equations with respect to interchange in the composition of the alternating maps. In this paragraph we generalize the results for alternating maps to general p -periodic non-autonomous systems.

Theorem 3.4. *Let $\mu \geq 2$ and let be the p -periodic family of maps with individual mappings f_0, f_1, \dots, f_{p-1} with $f_j \in C^{\mu+1}(I_j)$, with one fixed $j \in \{0, \dots, p-1\}$, and a periodic point $a_j \in I_j$, with period p , i.e., a fixed point of F_j , satisfying:*

- (1) *There exist a_0, a_1, \dots, a_{p-1} , fixed points of F_0, F_1, \dots, F_{p-1} , respectively, that is*

$$\begin{aligned} F_0(a_0) &= a_0, \\ F_1(a_1) &= a_1, \\ &\vdots \\ F_j(a_j) &= a_j, \\ &\vdots \\ F_{p-1}(a_{p-1}) &= a_{p-1}. \end{aligned}$$

- (2) *The non-hyperbolicity condition*

$$\partial_x F_j(x)|_{x=a_j} = \prod_{i=0}^{p-1} \partial_x f_i(a_j) = 1.$$

(3) *Higher degeneracy conditions*

$$\partial_{x^i} F_j(x)|_{x=a_j} = 0, \text{ for every } 2 \leq i \leq \mu.$$

(4) *The non-degeneracy condition*

$$\partial_{x^{\mu+1}} F_j(x)|_{x=a_j} \neq 0.$$

Then, all the compositions F_m , $0 \leq m \leq p-1$, satisfy

$$\partial_{x^i} F_m(x)|_{x=a_m} = 0, \text{ for every } 2 \leq i \leq \mu$$

and

$$\partial_{x^{\mu+1}} F_m(x)|_{x=a_m} \neq 0.$$

Proof. Without loss of generality we consider that the hypothesis apply when $j = 0$, what can be done re-indexing the maps of the p -periodic system. We now apply Theorem 3.2 to the alternating system $f_0, f_{p-1} \circ \dots \circ f_1$ with compositions $F = F_0$ and $G = F_1 = f_0 \circ F \circ f_0^{-1}$, making $a = a_0$, $b = a_1$ and getting the result for $j = 1$. Applying the same argument repeatedly, the result follows immediately for all the cyclic compositions F_m , $0 \leq m \leq p-1$. ■

Remark 3.5. *The same result holds for k -periodic points of the compositions F_j , i.e., $k \times p$ -periodic points of the alternating system, since in that case we apply Theorem 3.2 to the alternating system with compositions $F = F_0^k$ and $G = F_1^k = f_0 \circ F^k \circ f_0^{-1}$.*

After this result we can choose the composition order that makes the bifurcation equations easier to solve.

3.2. Invariance of the transversality conditions.

3.2.1. Alternating maps. In this paragraph we prove the symmetry for the transversality conditions concerning cyclic compositions of p maps. We follow the same technique of proving the result for the alternating maps² f and g and generalizing it to periodic points of p -periodic systems. Suppose that there exists the solution Λ_0 in the parameter space of the bifurcation equations 9 and that the non-degeneracy condition 10 holds at Λ_0 , that solution coexists with fixed points a and b for the compositions F and G .

Consider the map $\mathcal{F} = (F, F_x, F_{xx}, \dots, F_{x^{\mu-1}})$ with the derivatives of the composition F and the Jacobian determinant of \mathcal{F} , now as a function of Λ . It is the determinant

$$(23) \quad J_\Lambda \mathcal{F}(x, \Lambda) = \det \begin{bmatrix} F_{\lambda_1}(x, \Lambda) & \cdots & F_{\lambda_\mu}(x, \Lambda) \\ \vdots & \ddots & \vdots \\ F_{x^{\mu-1}\lambda_1}(x, \Lambda) & \cdots & F_{x^{\mu-1}\lambda_\mu}(x, \Lambda) \end{bmatrix}.$$

Consider similar definitions for $\mathcal{G}(x, \Lambda)$ and $J_\Lambda \mathcal{G}(x)$ relative to the composition G . With the previous definitions we can establish the next lemma.

Lemma 3.6. *The Jacobians $J_\Lambda \mathcal{F}(x, \Lambda)$ and $J_\Lambda \mathcal{G}(x, \Lambda)$ computed at the solutions of the bifurcation conditions (9) and (10) satisfy the equality*

$$(24) \quad J_\Lambda \mathcal{F}(a, \Lambda_0) = (f_x(a, \Lambda_0))^{\frac{3\mu-\mu^2}{2}} J_\Lambda \mathcal{G}(b, \Lambda_0).$$

²Notation that we adopt in this paragraph simplify the presentation of the next results and proofs, we replace f_0 by f and f_1 by g . The compositions are $F = g \circ f$ and $G = f \circ g$.

Proof. The proof rests on the fact that we can obtain the lines of the Jacobian matrix $[J_\Lambda \mathcal{F}(a, \Lambda_0)]$ of \mathcal{F} using Gaussian manipulation of the Jacobian matrix $[J_\Lambda \mathcal{G}(b, \Lambda_0)]$ of \mathcal{G} .

Consider

$$[J_\Lambda \mathcal{F}(a, \Lambda_0)] = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_\mu \end{bmatrix},$$

where L_i denotes the i line of the matrix. We have to prove that

$$[J_\Lambda \mathcal{G}(b, \Lambda_0)] = \begin{bmatrix} \alpha_{11} L_1 \\ \alpha_{21} L_1 + \alpha_{22} L_2 \\ \vdots \\ \sum_{j=1}^{\mu} \alpha_{\mu j} L_j \end{bmatrix},$$

i.e.

$$[J_\Lambda \mathcal{G}(b, \Lambda_0)] = \begin{bmatrix} \alpha_{11} & 0 & \cdots & 0 \\ \alpha_{21} & \alpha_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{\mu 1} & \alpha_{\mu 2} & \cdots & \alpha_{\mu \mu} \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_\mu \end{bmatrix},$$

which is

$$[J_\Lambda \mathcal{G}(b, \Lambda_0)] = A [J_\Lambda \mathcal{F}(a, \Lambda_0)],$$

where

$$A = \begin{bmatrix} \alpha_{11} & 0 & \cdots & 0 \\ \alpha_{21} & \alpha_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{\mu 1} & \alpha_{\mu 2} & \cdots & \alpha_{\mu \mu} \end{bmatrix}$$

and the central point that $\det A$ must be different from 0.

The fact that for general $\mu \geq 1$ the matrix A is a lower triangular matrix is trivial. Bellow, we prove that each entry of the main diagonal is

$$(25) \quad \alpha_{jj} = (f_x(a, \Lambda_0))^{2-j}, \quad 1 \leq j \leq \mu,$$

this equality implies that all such entries are different from zero after the non-hyperbolicity condition at the bifurcation

$$\partial_x F(x, \Lambda)|_{x=a, \Lambda=\Lambda_0} = g_x(b, \Lambda_0) f_x(a, \Lambda_0) = 1.$$

Moreover, (25) implies that the determinant of A is

$$\det A = \prod_{j=1}^{\mu} (f_x(a, \Lambda_0))^{2-j} = (f_x(a, \Lambda_0))^{\frac{3\mu - \mu^2}{2}},$$

as desired.

We prove now equality (25). Note that $G \circ f = f \circ g \circ f = f \circ F$. We derive this local conjugacy in order to λ_i , with $i = 1, 2, \dots, \mu$. We have

$$(26) \quad \partial_{\lambda_i} G(f(x, \Lambda), \Lambda) = \partial_{\lambda_i} f(F(x, \Lambda), \Lambda),$$

with

$$(27) \quad \begin{aligned} \partial_{\lambda_i} G(f(x, \Lambda), \Lambda) &= G_{\lambda_i}(f(x, \Lambda), \Lambda) + G_x(x, \Lambda) f_{\lambda_i}(x, \Lambda), \\ \partial_{\lambda_i} f(F(x, \Lambda), \Lambda) &= f_{\lambda_i}(x, \Lambda) + f_x(F(x, \Lambda), \Lambda) F_{\lambda_i}(x, \Lambda). \end{aligned}$$

At the points (a, Λ_0) and (b, Λ_0) equating the second members of (27) one has

$$G_{\lambda_i}(b, \Lambda_0) + G_x(b, \Lambda_0) f_{\lambda_i}(a, \Lambda_0) = f_{\lambda_i}(a, \Lambda_0) + f_x(a, \Lambda_0) F_{\lambda_i}(a, \Lambda_0),$$

which after conditions (9) is at the bifurcation point

$$(28) \quad G_{\lambda_i}(b, \Lambda_0) = f_x(a, \Lambda_0) F_{\lambda_i}(a, \Lambda_0),$$

this equality gives the relation between the first rows of the Jacobians.

To get the relations between the second rows we consider the derivative relative to x of (27), we present only the terms that matter for the computation of the main diagonal of A

$$(29) \quad \begin{aligned} \partial_{\lambda_i x} G(f(x, \Lambda), \Lambda) &= G_{\lambda_i x}(f(x, \Lambda), \Lambda) f_x(x, \Lambda) + \dots \\ \partial_{\lambda_i x} f(F(x, \Lambda), \Lambda) &= \dots + f_x(F(x, \Lambda), \Lambda) F_{\lambda_i x}(x, \Lambda). \end{aligned}$$

which after conditions (9) gives at the bifurcation value equalizing the right hand sides of (29)

$$(30) \quad G_{x\lambda_i}(b, \Lambda_0) = l.o.t + F_{x\lambda_i}(a, \Lambda_0),$$

where *l.o.t* stands for “lower order terms” in terms of derivatives on the dynamical variable of G_{λ_i} and F_{λ_i} , terms that do not appear in the main diagonal of A . This expression gives the relation between the second rows of the Jacobians.

To get the relation between the third rows of the two Jacobians, we consider the derivative of (29) regarding x

$$\begin{aligned} \partial_{\lambda_i x^2} G(f(x, \Lambda), \Lambda) &= G_{\lambda_i x^2}(f(x, \Lambda), \Lambda) f_x^2(x, \Lambda) + \dots \\ \partial_{\lambda_i x^2} f(F(x, \Lambda), \Lambda) &= \dots + f_x(F(x, \Lambda), \Lambda) F_{\lambda_i x^2}(x, \Lambda), \end{aligned}$$

which after conditions(9) gives at the bifurcation value

$$G_{x^2\lambda_i}(b, \Lambda_0) f_x^2(a, \Lambda_0) = l.o.t + f_x(a, \Lambda_0) F_{x^2\lambda_i}(a, \Lambda_0),$$

repeating this process and using the Faà di Bruno Formula (13) and the bifurcation equations (9), knowing that the lower order terms in derivatives relative to x (order less than $j - 1$) do not contribute to the diagonal of A , we have for $1 \leq j \leq \mu$

$$G_{x^{j-1}\lambda_i}(b, \Lambda_0) (f_x(a, \Lambda_0))^{j-1} = l.o.t + f_x(a, \Lambda_0) F_{x^{j-1}\lambda_i}(a, \Lambda_0),$$

dividing by $(f_x(a, \Lambda_0))^{j-1}$, which can not be zero from the second line of equations (9), we obtain

$$G_{x^{j-1}\lambda_i}(b, \Lambda_0) = l.o.t + (f_x(a, \Lambda_0))^{2-j} F_{x^{j-1}\lambda_i}(a, \Lambda_0),$$

the desired result. ■

3.2.2. Cyclic composition of p maps. In the general setting of p -periodic maps and considering the last lemma, the transversality conditions for A_μ bifurcations of $k \times p$ periodic points of the first two possible compositions of maps are such that

$$(31) \quad J_\Lambda \mathcal{F}_0^k(a_0, \Lambda_0) \neq 0 \Rightarrow J_\Lambda \mathcal{F}_1^k(a_1, \Lambda_0) \neq 0,$$

where J_Λ was defined in (23), because F_0^k and F_1^k are the two compositions of alternating maps as we have seen in Remark 3.5. The generalization to periodic points of all the cyclic compositions of p -periodic maps poses no difficulties and the proof is obtained by repeated application of Lemma 3.6.

Theorem 3.7. *If one of the transversality conditions of the A_μ bifurcation for $k \times p$ -periodic orbits of p -periodic maps at $\Lambda = \Lambda_0$ is satisfied, say*

$$J_\Lambda \mathcal{F}_j^k(a_j, \Lambda_0) \neq 0$$

then it is satisfied for all the cyclic compositions of the individual maps, i.e.,

$$\begin{aligned} J_\Lambda \mathcal{F}_0^k(a_0, \Lambda_0) &\neq 0, \\ &\vdots \\ J_\Lambda \mathcal{F}_{p-1}^k(a_{p-1}, \Lambda_0) &\neq 0. \end{aligned}$$

3.3. Conclusion. The invariance of degeneracy and transversality conditions, imply that the bifurcation problem with μ degeneracy conditions on the iteration variable is independent on the choice of the cyclic order in the composition of the maps when the maps are sufficiently differentiable. This invariance is fundamental, since it means that we can define local bifurcations in a unique way for families of p -periodic maps using any of the compositions of the particular maps.

In particular, the bifurcation set in the parameter space is the same for all F_j^k .

The main conclusion of this study, is that it suffices to solve the bifurcation conditions applied to one of the F_j^k possible compositions to obtain the bifurcation set. The bifurcation is well defined using the bifurcation conditions on the composition families. Each fiber replicates the behavior of the others. Hence, the local bifurcations studied in this work of p -periodic difference equations are defined by the same rules of the local bifurcations of autonomous systems.

4. EXAMPLES

We conclude this work with the study of the particular case of alternating maps. We establish some useful criteria about the existence of the swallowtail singularity for alternating maps and give two examples exhibition this type of bifurcation.

Let $f \in \mathcal{C}^3(I)$. The Schwarzian derivative of f is

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left(\frac{f''(x)}{f'(x)} \right)^2,$$

defined for every x in I , that is not a critical point of f .

Proposition 4.1. *Consider an alternating system with families, $f = f_0$ and $g = f_1$, satisfying all the conditions of the A_3 bifurcation, i.e., swallowtail bifurcation, together with the transversality conditions. If one of the maps say, without loss of generality, g , has Schwarzian derivative different from zero at b , $Sg(b) \neq 0$,*

then the product of the Schwarzian derivatives must be negative at the swallowtail bifurcation point, i.e.,

$$Sg(b) \cdot Sf(a) < 0.$$

Proof. Recall the example 3.3 with the same notation for derivatives. Consider f and g in the conditions of that example. Remember the equalities obtained and the same simplifying notation. If g is assumed to have negative Schwarzian derivative at b , one has

$$\frac{g_3}{g_1} - \frac{3}{2} \left(\frac{g_2}{g_1} \right)^2 < 0,$$

then, from (16) and (21) one has

$$\frac{g_3}{g_1} = \frac{3(f_1)^2}{(f_1)^4} - \frac{f_3}{(f_1)^3} = \frac{1}{(f_1)^2} \left(3 \left(\frac{f_2}{f_1} \right)^2 - \frac{f_3}{f_1} \right) < \frac{3}{2} \left(\frac{g_2}{g_1} \right)^2,$$

and, by the equality (19)

$$\frac{g_2}{g_1} = -\frac{f_2}{(f_1)^2},$$

the inequality above becomes

$$3 \left(\frac{f_2}{f_1} \right)^2 - \frac{f_3}{f_1} < \frac{3}{2} \left(\frac{f_2}{f_1} \right)^2.$$

Therefore,

$$\frac{f_3}{f_1} - \frac{3}{2} \left(\frac{f_2}{f_1} \right)^2 > 0.$$

Therefore, if g has negative Schwarzian derivative at b , then f must have positive Schwarzian derivative at a .

On the other hand, if f is assumed to have negative Schwarzian derivative, then by similar reasonings

$$\frac{f_3}{f_1} = -\frac{g_3}{g_1} (f_1)^2 + 3 \left(\frac{f_2}{f_1} \right)^2 < \frac{3}{2} \left(\frac{f_2}{f_1} \right)^2.$$

This implies

$$\frac{g_3}{g_1} (f_1)^2 > \frac{3}{2} \left(\frac{f_2}{f_1} \right)^2$$

and, as

$$\frac{g_2}{g_1} = -\frac{f_2}{(f_1)^2},$$

it follows that

$$\frac{g_3}{g_1} > \frac{3}{2} \left(\frac{f_2}{(f_1)^2} \right)^2 = \frac{3}{2} \left(\frac{g_2}{g_1} \right)^2.$$

Hence, if f has negative Schwarzian derivative then g must have positive Schwarzian derivative. ■

As in [9], while working on pitchfork bifurcation, we can state the two following propositions for the A_3 degenerate bifurcation. The proofs are similar.

Proposition 4.2. *Let f and g be \mathcal{C}^3 alternating maps. If f is strictly increasing and g is strictly decreasing in x (analogously, if f is strictly decreasing and g is strictly increasing), then the alternating system associated with f and g cannot have a A_3 degenerate bifurcation.*

Proposition 4.3. *Let f and g be \mathcal{C}^3 alternating maps. If f and g are both strictly increasing in the dynamic variable and one of these two situations happens*

$$\min_{x \in I_0, I_1} (\partial_{x^2} f(x), \partial_{x^2} g(x)) > 0 \text{ (both convex)}$$

or

$$\max_{x \in I_0, I_1} (\partial_{x^2} f(x), \partial_{x^2} g(x)) < 0 \text{ (both concave)},$$

then the alternating system generated by them cannot have a swallowtail bifurcation.

Proposition 4.4. *If f and g are both strictly decreasing in the dynamic variable and this situation happens*

$$\max_{x \in I_0, I_1} (\partial_{x^2} f(x), \partial_{x^2} g(x)) > 0 \text{ and } \min_{x \in I_0, I_1} (\partial_{x^2} f(x), \partial_{x^2} g(x)) < 0,$$

i.e., one is concave and one is convex or vice-versa, then the alternating system generated by them cannot have a A_3 degenerate bifurcation.

At this point, we consider two concrete examples. The first one is an alternating system with polynomial families, and the second with the tangent family, $\lambda \tan x$, and a polynomial family. The first example is relatively easy to compute, but the second one has elusive roots due to its high degeneracy. Thus, some numeric work was necessary. We present only the solutions and discard the tedious computations of the second example.

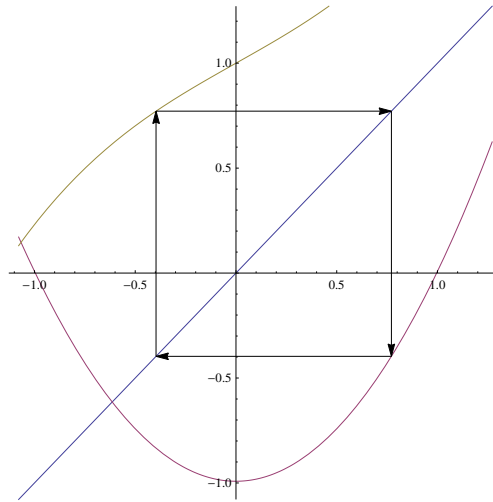


FIGURE 2. The geometry of the individual maps at the Swallowtail bifurcation point A_3 for example 4.5. Please note that one map is convex and the other is concave.

Example 4.5. Alternating system f and g with a quadratic polynomial $f_0 = x^2 + \lambda_1$ and a cubic polynomial $f_1 = \lambda_3 x^3 + \lambda_2 x + 1$, defined in the real line. The compositions are

$$F(x) = f_1 \circ f_0(x) = \lambda_3 x^6 + 3\lambda_3 \lambda_1 x^4 + (3\lambda_3 \lambda_1^2 + \lambda_2) x^2 + 1 + \lambda_2 \lambda_1 + \lambda_3 \lambda_1^3$$

and

$$G(x) = f_0 \circ f_1(x) = \lambda_3^2 x^6 + 2\lambda_2 \lambda_3 x^4 + 2\lambda_3 x^3 + \lambda_2^2 x^2 + 2\lambda_2 x + 1 + \lambda_1.$$

The bifurcation equations (9) for F or G have solutions

$$\lambda_1 = -\frac{3^5}{5 \cdot 7^2}, \lambda_2 = \frac{5^2 \cdot 7}{2^2 \cdot 3^4} \text{ and } \lambda_3 = \frac{7^5}{2^4 \cdot 3^{10}},$$

with

$$a = \frac{3^3}{5 \cdot 7}, b = -\frac{2 \cdot 3^5}{5^2 \cdot 7^2},$$

such that $f_0(a) = b$ and $f_1(b) = a$. The Schwarzian derivatives are

$$Sf(a) = \frac{1}{b} = -\frac{5^2 \cdot 7^2}{2 \cdot 3^5}, Sg(b) = \frac{1}{6} \frac{1}{b^2} = \frac{5^4 \cdot 7^4}{2^3 \cdot 3^{11}},$$

naturally, with opposite signs at the bifurcation points, accordingly with Proposition 4.1. Obviously, $SF(a) = SG(b) = 0$ at the bifurcation points.

The transversality condition (31) is from Lemma 3.6 equal for F and G (since $\frac{3 \cdot 3 - 3^2}{2} = 0$) and gives

$$J_\Lambda \mathcal{F}(a, \Lambda_0) = J_\Lambda \mathcal{G}(b, \Lambda_0) = -\frac{2^4 \cdot 3^{10}}{5^4 \cdot 7^3} \neq 0.$$

We can see at Figure 2 the geometry of the individual maps at the A_3 singularity. Both functions are increasing at suitable neighborhoods of a and b and one function is concave and the other is convex.

The bifurcation set is exactly similar to the one depicted in Figure 1. We have the same behavior of the two possible compositions F and G .

Example 4.6. Consider now the family of real alternating maps f_0 and f_1 with $f_0(x) = -x^4 + \lambda_1 x^2 + x + \lambda_2$ and $f_1(x) = \lambda_3 \tan x$, defined in suitable open sets near the solutions of the swallowtail bifurcation equations. We have the solutions of the bifurcation conditions $a \simeq 0.0797053$, $b \simeq 0.0793675$, $\lambda_1 \simeq -0.0400839$, $\lambda_2 \simeq -0.0000428492$, $\lambda_3 \simeq 1.00215$. The non-degeneracy condition gives $F_{x^4}(a) \simeq -26.7$. We note that a and b are very near each other and the maps are almost parallel at the bifurcation points, as we can see at Figure 3. The Schwarzian derivatives are

$$Sf(a) = -1.96648, Sg(b) = 2.$$

The transversality condition (31) is again equal for F and G and is

$$J_\Lambda \mathcal{F}(a, \Lambda_0) = J_\Lambda \mathcal{G}(b, \Lambda_0) = -2.08013.$$

These two examples, exhibiting the swallowtail bifurcation, produce evidence that high degeneracy bifurcations can occur in concrete examples and the theory is not void of applications.

Acknowledgement The author wants to thank the anonymous referee about his precious remarks and suggestions that improved a great deal the revised version of this paper. The author also thanks Michal Misiurewicz for the fruitful discussion

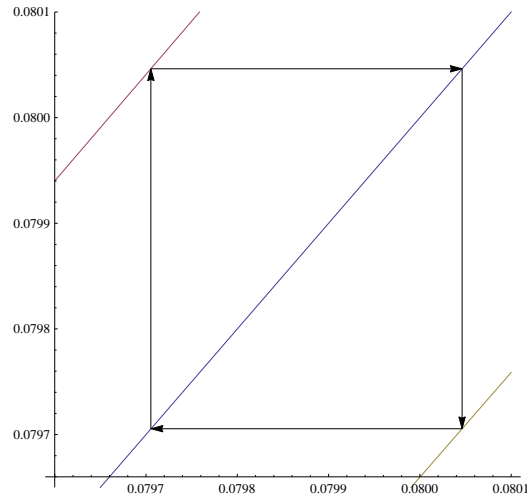


FIGURE 3. The orbit of the bifurcation points a and b of f_0, f_1 in example 4.6 viewed as a cobweb diagram. The maps, one concave and the other convex, are almost parallel.

in the Polish countryside about the proof of Lemma 3.1. The author was partially funded through project PEst-OE/EEI/LA0009/2013 for CAMGSD.

REFERENCES

- [1] D. J. Allwright, Hypergraphic functions and bifurcations in recurrence relations, *Siam Journal on Applied mathematics* 34 (4) (1978) 687–691.
- [2] J. F. Alves, L. Silva, Nonautonomous graphs and topological entropy of nonautonomous lorenz systems, *International Journal of Bifurcation and Chaos* (To appear).
- [3] V. I. Arnold, Critical points of smooth functions, In *Proceedings of ICM 74* (1) (1979) 19–40.
- [4] V. I. Arnold, *Dynamical Systems. V. Bifurcation Theory and Catastrophe Theory*, *Encyclopedia of Mathematical Sciences*, vol. 5 of *Encyclopaedia of Mathematical Sciences*, Springer, Berlin, 1994.
- [5] B. Aulbach, M. Rasmussen, S. Siegmund, Approximation of attractors of nonautonomous dynamical systems, *Discrete and Continuous Dynamical Systems* 5 (2) (2005) 215–238.
- [6] W. Beyn, T. Hls, M. Samtenschnieder, On r -periodic orbits of k -periodic maps, *Journal of Difference Equations and Applications* 8 (14) (2008) 865–887.
- [7] K. Brucks, H. Bruin, *Topics from one-dimensional dynamics*, vol. 62, Cambridge University Press, 2004.
- [8] S. Chow, J. Hale, *Methods of bifurcation theory*, vol. 251, Springer, 1982.
- [9] E. D’Aniello, H. M. Oliveira, Pitchfork bifurcation for non-autonomous interval maps, *Difference Equations and Applications* 15 (3) (2009) 291–302.
- [10] W. de Melo, S. Strien, *One-dimensional dynamics*, Springer, Berlin, Heidelberg, 1993.
- [11] S. Elaydi, R. Luis, H. Oliveira, Local bifurcation in one dimensional non-autonomous periodic difference equations, *International Journal of Bifurcation and Chaos* 23 (3) (2013) 1–18.
- [12] S. Elaydi, R. Sacker, Skew-product dynamical systems: Applications to difference equations, in: *Proceedings of the Second Annual Celebration of Mathematics*, 2005.
- [13] R. Gilmore, *Catastrophe theory for scientists and engineers*, Dover Publications, 1993.
- [14] M. Golubitsky, D. Schaeffer, *Singularities and Groups in Bifurcation Theory*, vol. 51, *Applied Mathematical Sciences*, 1985.
- [15] J. Guckenheimer, On the bifurcation of maps of the interval, *Inventiones mathematicae* 39 (2) (1977) 165–178.

- [16] T. Hüls, A model function for non-autonomous bifurcations of maps, *Discrete and continuous dynamical systems series B* 7 (2) (2007) 351.
- [17] G. Iooss, *Bifurcations of maps and applications*, vol. 36, *Mathematics Studies*, (North-Holland, Amsterdam, New York, Oxford), France, 1979.
- [18] W. Johnson, The curious history of Faà di Bruno's formula, *The American mathematical monthly* 109 (3) (2002) 217–234.
- [19] P. Kloeden, C. Pötzsche, M. Rasmussen, *Discrete time nonautonomous dynamical systems*, Manuscript, 2011.
- [20] P. Kloeden, M. Rasmussen, *Nonautonomous dynamical systems*, *Mathematical Surveys and Monographs*, vol. 176, *Mathematical Surveys and Monographs*, 2011.
- [21] P. Kloeden, S. Siegmund, Bifurcations and continuous transitions of attractors in autonomous and nonautonomous systems, *International Journal of Bifurcation and Chaos* 15 (3) (2005) 743–762.
- [22] S. Kolyada, M. Misiurewicz, L. Snoha, Topological entropy of nonautonomous piecewise monotone dynamical systems on the interval, *Fundamenta Mathematicae* (160) (1997) 161–181.
- [23] I. A. Kuznetsov, *Elements of applied bifurcation theory*, vol. 112, 3rd ed., Springer, New York, Berlin, Heidelberg, 1998.
- [24] S.-F. Lacroix, *Traité du Calcul différentiel et du Calcul intégral*, 2e Édition Revue et Augmentée, vol. 1, Courcier, Paris, 1810.
- [25] S. Noschese, P. Ricci, Differentiation of multivariable composite functions and bell polynomials, *Computational Analysis and Applications* 5 (3) (2003) 333–340.
- [26] C. Pötzsche, *Geometric theory of discrete nonautonomous dynamical systems*, Springer, Berlin, 2010.
- [27] C. Pötzsche, Nonautonomous bifurcation of bounded solutions I: A Lyapunov-Schmidt approach, *Discrete and Continuous Dynamical Systems, series B* 14 (2) (2010) 739–776.
- [28] C. Pötzsche, Bifurcations in a periodic discrete-time environment, *Real World Applications* 14 (1) (2013) 53–82.
- [29] C. Pötzsche, Nonautonomous bifurcation of bounded solutions II: A shovel bifurcation pattern, *Discrete and Continuous Dynamical Systems, series A* 31 (3) (2013) 941–973.
- [30] M. Rasmussen, Towards a bifurcation theory for nonautonomous difference equations, *Difference Equations and Applications* 12 (3-4) (2006) 297–312.
- [31] M. Rasmussen, *Attractivity and bifurcation for nonautonomous dynamical systems*, vol. 1907, Springer, Berlin Heidelberg, 2007.
- [32] S. Roman, The formula of Faà di Bruno, *The American Mathematical Monthly* 87 (10) (1980) 805–809.
- [33] A. Sharkovsky, I. Maistrenko, E. Romanenko, *Difference equations and their applications*, vol. 250, Springer, Berlin Heidelberg, 1993.
- [34] D. Singer, Stable orbits and bifurcation of maps of the interval, *Applied Mathematics* 35 (2) (1978) 260–267.
- [35] R. Thom, *Stabilité structurelle et morphogénèse*, Interéditions, 1977.
- [36] E. Zeeman, *Catastrophe theory*, Addison-Wesley, 1977, selected papers.

CENTER FOR MATHEMATICAL ANALYSIS, GEOMETRY AND DYNAMICAL SYSTEMS, MATHEMATICS DEPARTMENT, INSTITUTO SUPERIOR TÉCNICO, UNIVERSIDADE DE LISBOA, AV. ROVISCO PAIS, 1049-001 LISBOA, PORTUGAL

E-mail address: holiv@math.ist.ulisboa.pt